RECENT PROGRESS OF NON COMMUTATIVE DIMENSION THEORY(Profound development of Operator Algebras)

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M.A. Rieffel [Rf] initiated stable rank of C^*-algebras which is considered as complex dimension of non commutative topological spaces. Successively, L.G. Brown and G.K. Pedersen [BP] introduced real rank of C^*-algebras, i.e. non commutative real dimension. These ranks are recently regarded as one of important indices for mild classification of C^*-algebras, in particular, simple C^*-algebras.

In [Rf], Rieffel proposed a problem such as describing stable rank of group C^*-algebras of Lie groups in terms of groups. For this problem, H. Takai and the author [ST1], [ST2] studied stable rank of group C^*-algebras of connected, solvable Lie groups of type I. The author [Sd1], [Sd2] extended partially their results to the case of amenable Lie groups of type I, and also considered the case of non amenable Lie groups of type I.

This talk is organized as follows: First of all, we review classes and examples of connected Lie groups, and some formulas of stable rank of group C^*-algebras of type I. Secondly, we give some new results for stable rank of the C^*-algebras of certain connected Lie groups of type I. Finally, we give some tables of both stable and real rank for some classes of C^*-algebras which includes some very important examples.

Definition. For a unital C^*-algebra $\mathfrak{A}$, its stable rank $\mathrm{sr}(\mathfrak{A})$ is defined by

$$\min\{n \in \mathbb{N} \mid L_n(\mathfrak{A}) \text{ is dense in } \mathfrak{A}^n\} \wedge \infty$$

where $\wedge$ means minimum and $a = (a_i)_{i=1}^n \in L_n(\mathfrak{A})$ means that $\sum_{i=1}^n a_i a_i^*$ is invertible in $\mathfrak{A}$. We define real rank $\mathrm{rr}(\mathfrak{A})$ of $\mathfrak{A}$ by

$$\min\{n-1 \in \{0\} \cup \mathbb{N} \mid L_n(\mathfrak{A}_{sa}) \text{ is dense in } (\mathfrak{A}_{sa})^n\} \wedge \infty$$

where $\mathfrak{A}_{sa}$ is the set of all self-adjoint elements of $\mathfrak{A}$. For a non unital C^*-algebra $\mathfrak{A}$, we define $\mathrm{sr}(\mathfrak{A}) = \mathrm{sr}(\mathfrak{A}^+)$, $\mathrm{rr}(\mathfrak{A}) = \mathrm{rr}(\mathfrak{A}^+)$ with $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$.

We give some important examples of connected Lie group as follows:

Table of Connected Lie Groups

Classes	Type I	Non Type I
Compact	$SO(n)$	None
Commutative	$\mathbb{R}^n \times \mathbb{T}^k$	None
Nilpotent	Heisenberg group $\mathbb{R}^2 \rtimes \mathbb{R}$	None
Solvable	$ax+b$ group $G = \mathbb{R} \rtimes \mathbb{R}$	Mautner group $\mathbb{R}^4 \rtimes \mathbb{R}$
Amenable	Motion groups $\mathbb{R}^n \rtimes SO(n)$	$(\mathbb{R}^4 \rtimes \mathbb{R}) \times SO(n)$
Semi-simple	$SL_n(\mathbb{R})$ $(n \geq 2)$	None
Reductive	$GL_n(\mathbb{R})$ $(n \geq 2)$	None
Non Amenable	$G \times SL_n(\mathbb{R})$ $(n \geq 2)$	$(\mathbb{R}^4 \rtimes \mathbb{R}) \times SL_n(\mathbb{R})$

where lower classes in each perpendicular section are wider than upper classes except for "Compact".

Notations. We denote by $\vee$ maximum.

For a topological space X, we let $\dim_{\mathbb{C}} X = [\dim X/2] + 1$ where $[\cdot]$ is the Gauss symbol.

For a Lie group G, denote by $[G, G]$ its commutator subgroup and by $\hat{G}_1$ the space of all 1-dimensional representations of G, and let Z the center of G. Let $C^*(G), C_r^*(G)$ be the full, reduced group C^*-algebra of G respectively.

Denote by $\mathrm{rr}(G)$ the real rank of a semi-simple Lie group G.

Table of stable rank of group C^*-algebras of type I

Classes	Stable rank
Compact	1
Commutative	$\dim_{\mathbb{C}} \hat{G}_1$
Simply connected, nilpotent	$\dim_{\mathbb{C}} \hat{G}_1$
Simply connected, solvable	$(2 \vee \dim_{\mathbb{C}} \hat{G}_1) \wedge \dim G$
Amenable	$\dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 2 \vee \dim_{\mathbb{C}} \hat{G}_1$
Semi-simple	$2 \wedge \mathrm{rr}(G)$
Reductive	$2 \wedge (\mathrm{rr}([G,G]) \vee (\dim \hat{Z} + 1))$
Non Amenable	$2 \wedge \mathrm{rr}(S) \leq \mathrm{sr}(C_r^*(G)) \leq 2$

where S is the quotient semi-simple Lie group of G by its radical, and lower classes in each perpendicular section are wider than upper classes except for "Compact".

Remark. If G is the generalized motion group, then $\mathrm{sr}(C^*(G)) = 1 = \dim_{\mathbb{C}} \hat{G}_1$.

If G is the direct product of the real $ax+b$ group and $SL_n(\mathbb{R})$, then $\mathrm{sr}(C_r^*(G)) = 2$.

New viewpoint. *Let G be a connected Lie group of type I. Then*

$$\dim_{\mathbb{C}} \hat{G}_{r,1} \leq \mathrm{sr}(C_r^*(G)) \leq 2 \vee \dim_{\mathbb{C}} \hat{G}_{r,1}$$

where $\hat{G}_{r,1}$ is the space of all 1-dimensional representations in the reduced dual $\hat{G}_r$.

Lemma 1 [ST2]. *Let G be a simply connected, solvable Lie group. Then* $\mathrm{sr}(C^*(G)) = 1$ *if and only if $G \cong \mathbb{R}$.*

Remark. In the proof of Lemma 1, we showed that for a crossed product of the form $\mathfrak{A} = C_0(\mathbb{R}^n) \rtimes \mathbb{R}$, $\mathrm{sr}(\mathfrak{A}) = 1$ if and only if $\mathfrak{A} = C_0(\mathbb{R})$.

Lemma 2. *Let G be a connected solvable Lie group, and $\tilde{G}$ its universal covering group. If the center Z of $\tilde{G}$ is connected, then* $\mathrm{sr}(C^*(G)) = 1$ *if and only if G is isomorphic to $\mathbb{R}$ or $\mathbb{T}^s$ or $\mathbb{T}^s \times \mathbb{R}$.*

Proof. If G is commutative, then $G \cong \mathbb{R}^k \times \mathbb{T}^s$ for some k, s. Then $\mathrm{sr}(C^*(G)) = 1$ if and only if $G \cong \mathbb{R}$ or $\mathbb{T}^s$ or $\mathbb{R} \times \mathbb{T}^s$.

Suppose that G is non commutative. Let Γ be a discrete central normal subgroup of $\tilde{G}$ such that $\tilde{G}/\Gamma \cong G$. By the third homomorphism theorem, $\tilde{G}/Z \cong (\tilde{G}/\Gamma)/(Z/\Gamma) \cong G/(Z/\Gamma)$. Hence, there exists a surjective $*$-homomorphism from $C^*(G)$ to $C^*(\tilde{G}/Z)$. Since Z is connected, $\tilde{G}/Z$ is a simply connected, solvable Lie group by homotopy exact sequence. By [Lemma 1], one has that $\mathrm{sr}(C^*(\tilde{G}/Z)) = 1$ if and only if $\tilde{G}/Z \cong \mathbb{R}$. In this case, $\tilde{G} \cong Z \rtimes \mathbb{R} \cong \mathbb{R}^k \rtimes \mathbb{R}$ for some k. Then $G \cong (\mathbb{R}^{k-s} \times \mathbb{T}^s) \rtimes \mathbb{R}$ for some s since Γ is central. It follows that $C^*(G) \cong C_0(\mathbb{R}^{k-s} \times \mathbb{Z}^s) \rtimes \mathbb{R}$. Since $\mathbb{R}^{k-s} \times \{0\}$ is invariant under the action of $\mathbb{R}$, and closed in $\mathbb{R}^{k-s} \times \mathbb{Z}^s$, then $C_0(\mathbb{R}^{k-s}) \rtimes \mathbb{R}$ is a quotient C^*-algebra of $C^*(G)$. If $k - s \geq 1$, then $\mathrm{sr}(C_0(\mathbb{R}^{k-s}) \rtimes \mathbb{R}) \geq 2$ by [Remark of Lemma 1]. If $k - s = 0$, then $C^*(G) \cong \oplus_{\mathbb{Z}^s} C^*(\mathbb{R})$, which is commutative. $\square$

Remark. If $G = \mathbb{R}^2 \rtimes_\beta \mathbb{R}$ where β is rotation on $\mathbb{R}^2$, then its center is isomorphic to $\mathbb{Z}$. This example is the non exponential, simply connected, solvable Lie group unique up to isomorphisms with dimension ≤ 3.

It is known that connected is the center of any connected, nilpotent Lie group.

Corollary 3. *Let G be a connected nilpotent Lie group. Then the following are equivalent:*

(1) $\mathrm{sr}(C^*(G)) = 1$.

(2) *G is isomorphic to either $\mathbb{R} \times \mathbb{T}^k$ or $\mathbb{R}$ or $\mathbb{T}^k$.*

(3) $\dim_{\mathbb{C}} \hat{G}_1 = 1$.

Sketch of Proof. The equivalence of (1) and (2) are obtained by Lemma 2 and the fact that connected is the center of any connected, nilpotent Lie group.

The implication (2) $\Rightarrow$ (3) is trivial. But the converse is non trivial. We must consider the structure of G. But we omit it. $\square$

Theorem 4. *Let G be a connected nilpotent Lie group. Then*

$$\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} \hat{G}_1.$$

Remark. This is a generalization of the main theorem in [ST1] which states that the above equality holds for any simply connected, nilpotent Lie group.

Using the inequality in the amenable class in the table of stable rank of group C^*-algebras of type I, and by Lemma 2, we obtain the following:

Theorem 5. *Let G be a connected, solvable Lie group of type I. If the center of $\tilde{G}$ is connected, then*

$$\mathrm{sr}(C^*(G)) = \begin{cases} 1 & \textit{if } G \cong \mathbb{R} \textit{ or } \mathbb{T}^s \textit{ or } \mathbb{R} \times \mathbb{T}^s \\ 2 \vee \dim_{\mathbb{C}} \hat{G}_1 & \textit{otherwise.} \end{cases}$$

Problem. *If G is a simply connected, solvable Lie group of non type I, then* $\mathrm{sr}(C^*(G)) =$?

In this case, one can show that $\mathrm{sr}(C^*(G)) \geq \dim_{\mathbb{C}} \hat{G}_1$.

Example 6. If G is the Mautner group, then $\mathrm{sr}(C^*(G)) = 2 \geq 1 = \dim_{\mathbb{C}} \hat{G}_1$. Moreover, one has $\mathrm{sr}(C^*(G \times K)) = 2$ for any compact group K, and $\mathrm{sr}(C_r^*(G \times SL_n(\mathbb{R}))) = 2$.

As another example of non type I, let G be the Dixmier group which is the semi-direct product $\mathbb{R}^4 \rtimes H$ where H is the real Heisenberg group. Then $\mathrm{sr}(C^*(G)) = 2 = \dim_{\mathbb{C}} \hat{G}_1$.

Table of Discrete Groups

Classes	Type I	Non Type I
Amenable	$(\mathbb{Z}^n \times \mathbb{Z}_m) \rtimes F$, $(\lvert F\rvert < \infty)$	Heisenberg groups $\mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n$ $(n \geq 1)$ $(\mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n) \rtimes F$
Non Amenable	None	Free groups F_n $(n \geq 2)$ Free products $G_1 * G_2$ $SL_2(\mathbb{Z})$, Amalgams $G_1 *_H G_2$ $SL_n(\mathbb{Z})$ $(n \geq 3)$

where G_1, G_2, H are countable discrete groups.

Problem. *Let G be a discrete group. Then* $\mathrm{sr}(C_r^*(G)) =?$

Example 7. Let $G = \mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n$ be the generalized, discrete Heisenberg group. Then $G/[G,G] \cong \mathbb{Z}^{2n}$. It follows that $C^*(\mathbb{Z}^{2n})$ is a quotient C^*-algebra of $C^*(G)$. Hence $\mathrm{sr}(C^*(G)) \geq \dim_{\mathbb{C}} \mathbb{T}^{2n} = n+1$.

Dykema-Haagerup-Rørdam [DHR] showed that $\mathrm{sr}(C_r^*(G_1 * G_2)) = 1$ for discrete groups G_i with $|G_1| \geq 2$ and $|G_2| \geq 3$. In particular, since the free groups F_n $(n \geq 2)$ is isomorphic to $\mathbb{Z} * \cdots * \mathbb{Z}$ (n times), $\mathrm{sr}(C_r^*(F_n)) = 1$. Since $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, then $\mathrm{sr}(C_r^*(PSL_2(\mathbb{Z}))) = 1$.

On the other hand, $SL_2(\mathbb{Z})$ is isomorphic to the amalgam $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ where $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are respectively generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Moreover, $SL_3(\mathbb{Z})$ is not an amalgam, i.e. not isomorphic to $G_1 *_H G_2$.

Nagisa [Ng] showed that $\mathrm{sr}(C^*(\mathbb{Z}_m * \mathbb{Z}_n)) = \infty$ for $2 \leq m, n \leq \infty$, $m + n > 4$ and $\mathrm{sr}(C^*(\mathbb{Z}_2 * \mathbb{Z}_2)) = 1$.

Notations. Denote by $\mathbb{K}$ the C^*-algebra of all compact operators on a Hilbert spaces and by $\mathbb{B}$ the C^*-algebra of all bounded operators on a ∞-dimensional Hilbert space.

Table of Stable Rank

sr	Nuclear	Non Nuclear
1	$\mathbb{K}$, $C(S^1)$, AF, $\mathfrak{A}_\theta$, $\mathfrak{B}$, AT	$C_r^*(F_n)$ $(n \geq 2)$
2	$\mathbb{K} \otimes O_n$ $(n \geq 2)$, $C(S^i)$ $(i = 2, 3)$	$C^*(F_n) \otimes \mathbb{K}$ $(n \geq 2)$
$\vdots$	$C(S^n)$ $(n \geq 4)$	$C_r^*(F_2) \oplus C(S^n)$ $(n \geq 4)$
∞	$C([0,1]^\infty)$, O_n $(n \geq 2)$	$C^*(F_n)$ $(n \geq 2)$, $\mathbb{B}$

where $\mathfrak{B}$ is the Bunce-Deddens algebra and $\mathfrak{A}_\theta$ is the irrational rotation algebra. It is known that they are AT-algebras, i.e. inductive limits of the form $\varinjlim \oplus_{k=1}^{r_m} M_{n_k}(C(\mathbb{T}))$.

Table of Real Rank

rr	Nuclear	Non Nuclear
0	$\mathbb{K}$, AF, $\mathfrak{A}_\theta$, $\mathfrak{B}$, O_n $(n \geq 2)$	$O_n \otimes C_r^*(F_m)$, $\mathbb{B}$
1	$C(S^1)$, $C([0,1]) \otimes O_n \otimes \mathbb{K}$	$C_r^*(F_n)$ $(n \geq 2)$
$\vdots$	$C(S^n)$ $(n \geq 2)$	$C_r^*(F_2) \oplus C(S^n)$ $(n \geq 2)$
∞	$C([0,1]^\infty)$	$C^*(F_n)$ $(n \geq 2)$

Remark. Choi-Elliott [CE] proved that $\mathrm{rr}(\mathfrak{A}_\theta) = 0$. Blackadar and Kumjian showed that the Bunce-Deddens algebras $\mathfrak{B}$ of type 2^∞ have real rank zero. Nagisa [Ng] showed that $\mathrm{rr}(C^*(F_n)) = \infty$. By Nagisa-Osaka-Phillips, $\mathrm{rr}(C([0,1]) \otimes \mathfrak{A}) \geq 1$ for any C^*-algebra $\mathfrak{A}$. Beggs-Evans [BE] proved $\mathrm{rr}(\mathfrak{A} \otimes \mathbb{K}) \leq 1$.

Table of stable rank of simple C^*-algebras

sr	Nuclear	Non Nuclear
1	$\mathbb{K}$, $\mathfrak{A}_\theta$, $\mathfrak{B}$, $\mathcal{BL}$	$C_r^*(F_n)$ $(n \geq 2)$
2	$\mathbb{K} \otimes O_n$ $(n \geq 2)$, $\mathcal{VL}_2$	$\mathbb{K} \otimes O_n \otimes C_r^*(F_m)$ $(n, m \geq 2)$
$\vdots$	$\mathcal{VL}_k$ $(3 \leq k < \infty)$	?
∞	O_n $(n \geq 2)$	$O_n \otimes C_r^*(F_m)$, $\mathbb{B}/\mathbb{K}$

where $\mathcal{BL}$ is the Blackadar's simple unital projectionless C^*-algebra which is the inductive limit of mapping cones associated with an AF-algebra, and $\mathcal{VL}_k$ is the Villadsen's simple unital AH-algebra with stable rank k $(2 \leq k < \infty)$ [Vl].

Table of stable rank of finite simple unital C^*-algebras

sr	Nuclear	Non Nuclear
1	$\mathfrak{A}_\theta$, $\mathfrak{B}$, $\mathcal{BL}$, $\mathfrak{A}_\theta \otimes M_{2^\infty}$	$C_r^*(F_n)$ $(n \geq 2)$
$\vdots$	$\mathcal{VL}_k$ $(2 \leq k < \infty)$	?
∞	?	?

Remark. For any simple and infinite C^*-algebra $\mathfrak{A}$, we have $\mathrm{sr}(\mathfrak{A}) = \infty$. Rørdam [Rd] showed that if $\mathfrak{A}$ is simple and stably finite, and $\mathfrak{D}$ is a UHF-algebra, then $\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{D}) = 1$.

Table of real rank of simple C^*-algebras

rr	Nuclear	Non Nuclear
0	$\mathbb{K}$, $\mathfrak{A}_\theta$, $\mathfrak{B}$, O_n $(n \geq 2)$	$O_n \otimes C_r^*(F_m)$, $\mathbb{B}/\mathbb{K}$
1	$\mathcal{BL}$	$C_r^*(F_n)$ $(n \geq 2)$
$\vdots$	?	?
∞	?	?

Remark. Any simple, purely infinite C^*-algebra has real rank zero. Lin and Zhang constructed some simple C^*-algebras $\mathfrak{A}$ with $\mathrm{rr}(\mathfrak{A}) \neq 0$ and $\mathrm{rr}(M(\mathfrak{A})/\mathfrak{A}) = 0$, where $M(\mathfrak{A})$ is the multiplier of $\mathfrak{A}$. In fact, $\mathfrak{A}$ is a hereditary C^*-subalgebra of the tensor product of an AH-algebra and a UHF-algebra.

An AH-algebra $\mathfrak{A}$ is the inductive limit of the form $\varinjlim(\mathfrak{A}_n, \Phi_{m,n})$ where $\mathfrak{A}_n = \oplus_{j=1}^{r_n} C(\Omega_{n,j}, M_{[n,j]})$ with $\Omega_{n,j}$ connected, compact T_2-spaces, and $\Phi_{m,n} : \mathfrak{A}_n \to \mathfrak{A}_m$ $(m \geq n)$ unital homomorphisms. If $\sup_{n,j} \dim \Omega_{n,j} < \infty$, we call $\mathfrak{A}$ of bounded dimension. We say that $\mathfrak{A}$ has slow dimension growth if $\lim_{n\to\infty} \max_j (\dim \Omega_{n,j}/[n,j]) = 0$. If $\mathfrak{A}$ is simple, then $\lim_{n\to\infty} \min_j [n,j] = \infty$, and so $\mathfrak{A}$ has slow dimension growth if

$\sup \dim \Omega_{n,j} < \infty$. Note that if $\mathfrak{A}$ is simple, $\dim \Omega_{n,j} < \infty$ and $\mathfrak{D}$ is a UHF-algebra, then $\mathfrak{A} \otimes \mathfrak{D}$ is written as an inductive limit with slow dimension growth.

Table of stable rank of simple AH-algebras

sr	Classes
1	Bounded dimension Slow dimension growth
$\vdots$	$\mathcal{VL}_k$ $(2 \leq k < \infty)$
∞	?

Remark. Dadarlat-Nagy-Nemethi-Pasnicu [DNNP] showed the above case of bounded dimension. The case of slow dimension growth is proved by Blackadar-Dadarlat-Rørdam [BDR].

A C^*-algebra $\mathfrak{A}$ has the property (STP) (separating traces by projections) if equal are any two traces τ_i of $\mathfrak{A}$ satisfying $\tau_1(p) = \tau_2(p)$ for every projection $p \in \mathfrak{A}$.

Table of real rank of simple AH-algebras

rr	Classes
0	Simple AT + (STP) Slow dimension growth + (STP)
$\vdots$	?
∞	?

Remark. Blackadar-Bratteli-Elliott-Kumjian [BBEK] showed the case of simple AT-algebras. The case of slow dimension growth is obtained by Blackadar-Dadarlat-Rørdam [BDR].

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